2019
MATHEMATICS
Full Marks : 100
Pass Marks : 33
Time : Three hours
Attempt all Questions.

The figures in the right margin indicate full marks for the questions.

For Question Nos. 1 – 6, write the letter associated with the correct answer.

1. The value of $\sin \left[ \frac{\pi}{3} - \sin^{-1} \left( -\frac{1}{2} \right) \right]$ is:
   
   A. 1
   
   B. $\frac{1}{2}$
   
   C. $\frac{1}{\sqrt{2}}$
   
   D. 0

2. If $f : R \to R$ be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $f^{-1}(x)$ equals:
   
   A. $x^3$
   
   B. $\frac{1}{x^3}$
   
   C. $3 - x^3$
   
   D. $(3 - x^3)^{\frac{1}{3}}$

PT.O.
3. Mean and variance of a binomial distribution are 12 and 3 respectively. Then the number of trials is:
   
   A. 12  
   B. 15  
   C. 16  
   D. 36

4. \[ \int e^x \sec x (1 + \tan x) \, dx \] equals:

   A. \( e^x \cos x + C \)  
   B. \( e^x \sec x + C \)  
   C. \( e^x \sin x + C \)  
   D. \( e^x \tan x + C \)

5. The slope of the normal to the curve \( y = 2x^2 + 3 \sin x \) at \( x = 0 \) is:

   A. 3  
   B. \( \frac{1}{3} \)  
   C. -3  
   D. \( \frac{1}{3} \)
6. If the line \( \mathbf{r} = (-2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(-K\hat{i} + 2\hat{j} + \hat{k}) \) is parallel to the plane \( \mathbf{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) + 7 = 0 \), then the value of \( K \) is:

A. \( 0 \)
B. \( 1 \)
C. \( -1 \)
D. \( -2 \)

7. Show that the operation * on \( \mathbb{R}_+ \) (set of all positive real numbers) defined by \( a * b = \frac{ab}{3} \), \( \forall a, b \in \mathbb{R}_+ \) is a binary operation on \( \mathbb{R}_+ \).

8. Is Rolle's Theorem applicable to the function \( f(x) = |x| \) in the interval \([-1, 1]\) ?

9. If \( \frac{dy}{dx} = \frac{y}{x} \), prove that \( \frac{d^2y}{dx^2} = 0 \).

10. Prove that the function given by \( f(x) = x^3 - 3x^2 + 3x - 5 \) is increasing in \( \mathbb{R} \).

11. Evaluate: \( \int_{1}^{\sqrt{3}} \frac{1}{1 + x^2} \, dx \).

12. What is meant by the general solution of a differential equation?

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P.T.O.
13. If \( |\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}| \), find the angle between \( \vec{a} \) and \( \vec{b} \).


15. The cartesian equation of a line is \( \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \). Write its vector form.

16. If \( \alpha, \beta, \gamma \) be the angles made by a line with the coordinate axes, prove that 
\[ \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2. \]

17. Show that the relation \( R \) on \( \mathbb{N} \times \mathbb{N} \) defined by 
\( (a,b) R (c,d) \iff a + d = b + c, \ \forall \ (a,b), (c,d) \in \mathbb{N} \times \mathbb{N} \) 

is an equivalence relation.

18. If \( A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix} \) and \( I \) is the identity matrix of order 2, show that 
\( I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \).

19. Evaluate \( \int_{0}^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \, dx \).

20. Prove that \( \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + C \).

21. Find the differential equation of the family of curves \( y = e^x (A \cos x + B \sin x) \), where \( A \) and \( B \) are arbitrary constants.

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Contd.
22. Two cards are drawn simultaneously from a well shuffled pack of 52 cards.
Find the probability distribution of the number of aces.

23. Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right), (xy < 1)$ and hence deduce that

(i) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right), (xy > -1)$

(ii) $2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right), (|x| < 1)$

24. If the inverse of a square matrix exists, prove that it is unique. If $A$ and $B$ are both invertible square matrices of the same order, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

25. If $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{If } x < 0 \\ c, & \text{If } x = 0 \\ \sqrt{x + bx^2} - \sqrt{x} \times \frac{1}{bx^{3/2}}, & \text{If } x > 0 \end{cases}$$

is continuous at $x = 0$, find the values of $a$, $b$ and $c$.

26. Find $\frac{dy}{dx}$, if $x^y + y^x = a^b$.

**OR**

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P.T.O.
If \( x^y = e^{x-y} \), prove that \( \frac{dy}{dx} = \frac{\log x}{(\log ex)^2} \).

27. Find the area of the region bounded by the triangle whose vertices are \((-1, 2), (1, 5)\) and \((3, 4)\).

OR

Find the area of the region bounded by the lines \( x + 2y = 2, y - x = 1 \) and \( 2x + y = 7 \).

28. Find the integrating factor of the linear equation \( \frac{dy}{dx} + Py = Q \) and hence obtain the general solution of the equation.

29. Define cross product of two vectors and give the geometrical interpretation of the cross product of two vectors. If \( \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \) and \( \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} \), obtain the algebraic formula for \( \vec{a} \times \vec{b} \).

30. Prove that:

\[
\int_0^\pi \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2}; \quad (a < 1)
\]

OR

\[
\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}.
\]

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31. State and prove Baye's Theorem.

32. Derive the vector equation of a line passing through a given point and parallel to a given vector and hence obtain the cartesian equation of the line.

**OR**

Derive the vector equation of a plane in the normal form and hence obtain the cartesian equation of the plane.

33. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is \[ \tan^{-1}\sqrt{2}. \]

**OR**

Prove that the curves \( y^2 = x \) and \( xy = k \) cut at right angle if \( 8k^2 = 1 \).

34. If \( A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \), find \( AB \) and hence solve the following system of linear equations:

\[
\begin{align*}
x - y &= 3 \\
2x + 3y + 4z &= 17 \\
y + 2z &= 7.
\end{align*}
\]
35. Two godowns A and B have a given storage capacity of 100 quintals and 50 quintals respectively. They supply grain to three ration shops D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from godowns to the shops are given in the table below:

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>Transportation cost per quintal (in rupees)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D</td>
<td>E</td>
</tr>
<tr>
<td>A</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

How should the supplies be transported in order that the transportation cost is minimum? Solve the problem graphically.