2023

MATHEMATICS

Full Marks : 100

Pass Marks : 33

Time : Three hours

Attempt all Questions.

The figures in the right margin indicate full marks for the questions.

For Question Nos. 1 – 4, write the letter associated with the correct answer.

1. The principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is 1

   A. $\frac{\pi}{4}$

   B. $-\frac{\pi}{4}$

   C. $\frac{3\pi}{4}$

   D. $\frac{5\pi}{4}$

2. The integral $\int \operatorname{cosec} x\, dx \cos$ equals 1

   A. $\log|\sec x|+c$

   B. $\log|\sin x|+c$

   C. $\log|\sec x+\tan x|+c$

   D. $\log|\operatorname{cosec} x-\cot x|+c$

3. A homogeneous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ can be solved by making the substitution 1

A. $y = vx$

B. $v = xy$

C. $x = vy$

D. $y = v$

4. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is 1

A. $\frac{3}{2}$

B. $-\frac{3}{2}$

C. $\frac{1}{2}$

D. $-\frac{1}{3}$

5. Define an equivalence relation. 1

6. What is the range of the principal value branch of the function $\tan^{-1}$? 1

7. Find the value of $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$ 1

8. If a matrix has 15 elements, what are the possible orders it can have? 1

9. For what value of k is the function f defined by 1

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

continuous at $x=0$?

10. Find $\dfrac{dy}{dx}$, if $2x+3y=\sin y$. 1

11. What is the value of $\int e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)dx$? 1

12. Find the integrating factor of the differential equation $x\dfrac{dy}{dx}-y=2x^2$. 1

13. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$. 1

14. If a line has direction ratios 2, 1, -2, determine its direction cosines. 1

15. Show that $3\sin^{-1}x=\sin^{-1}\left(3x-4x^3\right)$, $x\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$. 2

16. Prove that the inverse of a square matrix, if it exists, is unique. 2

17. Find the equation of the tangent to the curve $y=x^4-6x^3+13x^2-10x+5$ at the point (0, 5). 2

18. Find the integral $\int\dfrac{e^{2x}-1}{e^{2x}+1}dx$. 2

19. Find the general solution of the first order linear differential equation $\dfrac{dy}{dx}+Py=Q$. 2

20. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then show that $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. 2

21. The cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form. 2

22. Find the mean of the number obtained on a throw of an unbiased die. 2

23. A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is $\frac{1}{100}$. What is the probability that he will win a prize at least once ? 2

24. Show that the function $f : [-1, 1] \to R$ given by $f(x) = \frac{x}{x+2}$ is one-one. Find the inverse of the function $f : [-1, 1] \to$ Range f. 4

Or

Let $*$ be the binary operation on N given by $a * b =$ L.C.M. of a and b. Is $*$ commutative ? Is $*$ associative ? Find the identity of $*$ in N. Which elements of N are invertible for the operation ?

25. Express the matrix $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix. 4

26. Find $\frac{dy}{dx}$, if $x^y + y^x = 1$. 4

Or

Prove that the greatest integer function f given by $f(x) = [x], 0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

27. Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$. 4

28. Prove that $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if f is an even function and 0, if f is an odd function. 4

29. Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. 4

30. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs. 1000 is deposited with this bank. How much will it worth after 10 years ($e^{0.5} = 1.648$) ? 4

31. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear. 4

32. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of equations 6

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

33. Show that the semi-vertical angle of the right circular cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$. 6

Or

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

34. Find the integral $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$. 6

Or

Show that $\int_0^{\frac{\pi}{2}} \log \sin x \, dx = -\frac{\pi}{2} \log 2$.

35. Find the vector equation of a line passing through two points in the form $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$. Also, derive the cartesian form $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ from the vector form. 6

Or

Find the vector equation of a plane perpendicular to a given vector and passing through a given point in the form $(\vec{r} - \vec{a}).\ \vec{N} = 0$. Also, derive the cartesian form $A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$ from the vector form.

36. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs. 6/kg and F_2 costs Rs. 5/kg, determine how much of each type of fertiliser should beused so that nutrient requirements are met at a minimum cost. What is the minimum cost? 6

37. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that

(i) it was produced by A,

(ii) it was produced by B ? 6
